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ARTIFICIAL BOUNDARY CONDITIONS FOR A TRANSMISSION BOUNDARY-VALUE PROBLEM FOR THE TIME-HARMONIC MAXWELL EQUATIONS WITHOUT DISPLACEMENT CURRENTS

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Artificial Boundary Conditions for a Transmission Boundary - Value Problem for the Time - Harmonic Maxwell Equations without Displacement Currents

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We consider a transmission boundary - value problem for the time harmonic Maxwell equations without displacement currents. As transmission conditions we use the continuity of the tangential parts of the magnetic field H and the continuity of the normal components of the magnetization $B = \mu H$. This problem, which is posed over all $\mathbb{R}^3$, is then *restricted to a bounded domain by introducing artificial boundary conditions.*

We present uniqueness and existence proofs for this problem using an integral equation approach and compare the results with those obtained in the ubounded case.

1. Introduction

A large number of different problems in electrical enigneering lead to transmission boundary value problems for the time - harmonic Maxwell equations:

Consider a bounded domain of conductive material $G^E \subset \mathbb{R}^3$, which is surrounded by air. In $G^L = \mathbb{R}^3 \setminus \overline{G}^E$ ($\overline{G}^E$ denotes the closure of G^E) a given, time - harmonic current density $\tilde{J}_e(x,t) = J_e(x)\, e^{-i\omega t}$ induces electromegnetic fields in G^E (Fig. 1). We are interested in computing the current densities in G^E which are due to the induced fields.

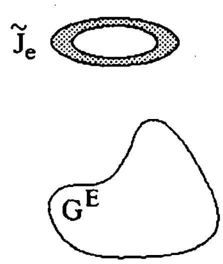

Fig. 1

The resulting *classical transmission boundary - value problem*

$$\begin{aligned} \operatorname{curl} H^L &= J_e - i\omega\varepsilon^L E^L \\ \operatorname{curl} E^L &= i\omega\mu^L H^L \end{aligned} \quad \text{in } G^L, \qquad \begin{aligned} \operatorname{curl} H^E &= (\sigma^E - i\omega\varepsilon^E)\, E^E \\ \operatorname{curl} E^E &= i\omega\mu^E H^E \end{aligned} \quad \text{in } G^E,$$

$$\begin{aligned} n \wedge H^E &= n \wedge H^L \\ n \wedge E^E &= n \wedge E^L \end{aligned} \quad \text{on } \Gamma = \partial G^E = \partial G^L,$$

with *Silver - Müller radiation condition*

$$H^L \wedge \frac{x}{|x|} - E^L = o(\frac{1}{|x|})$$

and coefficients

$\omega \geq 0$	frequency,
$\varepsilon^L, \varepsilon^E > 0$	electric permittivity in G^L, G^E,
$\mu^L, \mu^E > 0$	magnetic permeability in G^L, G^E,
$\sigma^E > 0$	electric conductivity in G^E,

is well investigated. Under certain regularity assumptions this problem is uniquely solvable [7, 10].

For devices working at low frequencies, the above problem is modified. The displacement currents are neglected, the boundary condition $n \wedge E^L = n \wedge E^E$ on Γ is changed to $n \cdot (\mu^E H^E) = n \cdot (\mu^L H^L)$ on Γ and the radiation condition is substituted by $H^L(x) = o(1)$, $E^L(x) = o(1)$ uniformly for $|x| \to \infty$. This new problem

$$\begin{aligned} \operatorname{curl} H^L &= J_e \\ \operatorname{curl} E^L &= i\omega\mu^L H^L \end{aligned} \quad \text{in } G^L, \qquad \begin{aligned} \operatorname{curl} H^E &= \sigma^E E^E \\ \operatorname{curl} E^E &= i\omega\mu^E H^E \end{aligned} \quad \text{in } G^E,$$

$$\begin{aligned} n \wedge H^E &= n \wedge H^L \\ n \cdot (\mu^E H^E) &= n \cdot (\mu^L H^L) \end{aligned} \quad \text{on } \Gamma. \tag{1}$$

$$H^L(x) = o(1), \qquad E^L(x) = o(1) \qquad \text{uniformly for } |x| \to \infty.$$

was investigated in [8].

For the application of certain numerical techniques (finite difference or finite volume schemes) to (1) G^L is cut off (Fig. 2). Instead of the unbounded domain G^L we now consider a bounded domain G_0^L. On the new boundary Γ_0, the values of $n \cdot (\mu^L H^L)$ are prescribed. The corresponding data is given by measurements or is estimated.

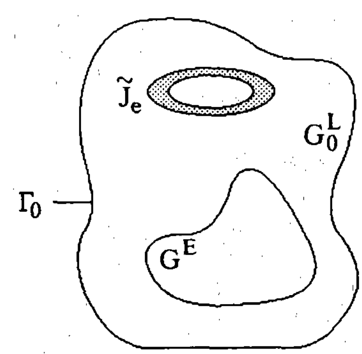

Fig. 2

In this paper, we consider (1) together with this artificial boundary condition:

$$\begin{aligned} \operatorname{curl} H^L &= J_e \\ \operatorname{curl} E^L &= i\omega\mu^L H^L \end{aligned} \quad \text{in } G_0^L, \qquad \begin{aligned} \operatorname{curl} H^E &= \sigma^E E^E \\ \operatorname{curl} E^E &= i\omega\mu^E H^E \end{aligned} \quad \text{in } G^E, \tag{2}$$

$$
\begin{aligned}
&n \wedge H^E = n \wedge H^L \\
&n \cdot (\mu^E H^E) = n \cdot (\mu^L H^L)
\end{aligned}
\qquad \text{on } \Gamma,
$$
$$
n \cdot (\mu^L H^L) = f \qquad \text{on } \Gamma_0. \tag{2}
$$

We show existence and uniqueness theorems for (2) and compare them with the corresponding results for (1).

2. Preliminaries

Before we start with the main part, we have to define the class of admissible domains G^E, G_0^L.

Let $C(G)$ $(C^k(G))$ denote the space of continuous (k times continuously differentiable) functions on G.

$G^E \subset \mathbb{R}^3$ is an open, bounded domain with C^2 boundary. The complement $G^L = \mathbb{R}^3 \setminus \overline{G}^E$ should be connected ($\overline{G}^E$ denotes the closure of G^E). G^E is the union of m connected components G_j^E, $j = 1,\dots,m$ having the topological genus p_j. The boundaries $\Gamma_j = \partial G_j^E$ are closed surfaces, which should be disjoint. Setting $\Gamma = \bigcup_{j=1}^{m} \Gamma_j$ we get $\Gamma = \partial G^E = \partial G^L$.

The topological genus of G^E resp. G^L is $p = \sum_{j=1}^{m} p_j$. There exist p surfaces $\Sigma_i^E \subset G^E$ resp. $\Sigma_i^L \subset G^L$, $i = 1,\dots,p$, such that $G^E \setminus \bigcup_{i=1}^{p} \Sigma_i^E$ resp. $G^L \setminus \bigcup_{i=1}^{p} \Sigma_i^L$ are simply connected. The boundary curves $\gamma_i^L = \partial \Sigma_i^E$ and $\gamma_i^E = \partial \Sigma_i^L$ lie on Γ.

Moreover, let G_0 be a simply connected, open, bounded domain in $\mathbb{R}^3$, such that

$$\overline{G}^E, \overline{\Sigma}_i^L \subset G_0.$$

The boundary $\Gamma_0 := \partial G_0$ is assumed to be C^2.

G_0^L is now defined as

$$G_0^L = G_0 \setminus \overline{G}^E.$$

Therefore, the topological genus of G_0^L is p and $G_0^L \setminus \bigcup_{i=1}^{p} \Sigma_i^L$ is simply connected

Example

Let G^E be a torus, G_0 be a sphere containing G^E in its interiour. In this case we have $m = p = 1$. The surfaces Σ_1^E, Σ_1^L and the curves γ_1^L, γ_1^E are shown in Fig. 3.

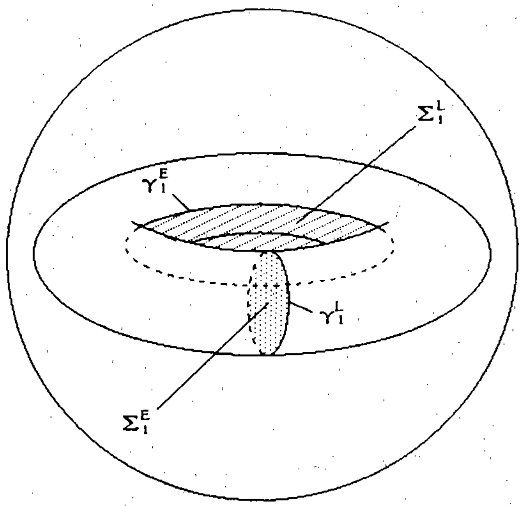

Fig. 3

Since G^E and G_0^L both have topological genus p, there exist p linear independent Neumann fields Z_i^E resp. Z_{0i}^L, $i = 1,\dots,p$, in G^E resp. G_0^L, fulfilling

$$\operatorname{curl} Z_i^E = 0, \quad \operatorname{div} Z_i^E = 0 \quad \text{in } G^E, \qquad n \cdot Z_i^E = 0 \quad \text{on } \Gamma = \partial G^E,$$

$$\operatorname{curl} Z_{0i}^L = 0, \quad \operatorname{div} Z_{0i}^L = 0, \quad \text{in } G_0^L, \qquad n \cdot Z_{0i}^L = 0 \quad \text{on } \Gamma \cup \Gamma_0 = \partial G_0^L,$$

$$\int_{\gamma_i^E} \tau \cdot Z_j^E \, dl = \delta_{ij}, \qquad \int_{\gamma_i^L} \tau \cdot Z_j^E \, dl = 0, \qquad \int_{\gamma_i^L} \tau \cdot Z_{0j}^L \, dl = \delta_{ij}, \qquad \int_{\gamma_i^E} \tau \cdot Z_{0j}^L \, dl = 0,$$

As a consequence of the regularity assumptions on G^E and G_0^L we get

$$Z_i^E \in C^\infty(G^E) \cap C^{0\alpha}(\overline{G}^E), \qquad Z_{0i}^L \in C^\infty(G_0^L) \cap C^{0\alpha}(\overline{G}_0^L).$$

For the prescribed data in (1) resp. (2) we suppose

$$J_e \in C^1(\mathbb{R}^3), \qquad \operatorname{div} J_e = 0, \qquad \operatorname{supp}(J_e) \subset G^J, \qquad \overline{G}^J \subset G_0^L \text{ bounded},$$

resp.

$$f \in C^{0\alpha}(\Gamma_0).$$

Moreover, we are looking for classical solutions of (2) satisfying

$$H^L, E^L \in C^1(G_0^L) \cap C(\overline{G}_0^L), \qquad H^E, E^E \in C^1(G^E) \cap C(\overline{G}^E).$$

In the subsequent analysis we make use of the following Banach spaces:

Let $0 < \alpha < 1$

- $C^{0\alpha}(G)$, $\|u\|_{0\alpha,G} = \sup\limits_{x \in G} |u(x)| + \sup\limits_{\substack{x \neq y \\ x,y \in G}} \dfrac{|u(x) - u(y)|}{|x - y|^\alpha}$,

  is the space of Hölder continuous functions on G.

- $C^{1\alpha}(G)$, $\|u\|_{1\alpha,G} = \sup_{x\in G}|u(x)| + \sup_{x\in G}|\nabla u(x)| + \sup_{\substack{x \neq y \\ x,y\in G}} \frac{|\nabla u(x) - \nabla u(y)|}{|x-y|^{\alpha}}$

  is the space of continuously differentiable functions on G with Hölder continuous derivatives.

- $V^{0\alpha}(\Gamma) = (C^{0\alpha}(\Gamma))^3$, $\|a\|_{V\alpha,\Gamma} = \max_{i=1,2,3}(\|a_i\|_{0\alpha,\Gamma})$,

  is the space of Hölder continuous vector fields on Γ.

- $T^{0\alpha}(\Gamma) = \{a \in V^{0\alpha}(\Gamma) \mid n\cdot a = 0\}$, $\|a\|_{T\alpha,\Gamma} = \|a\|_{V\alpha,\Gamma}$,

  is the space of Hölder continuous tangential fields on Γ.

- $T_d^{0\alpha}(\Gamma) = \{a \in T^{0\alpha}(\Gamma) \mid \mathrm{Div}\, a \in C^{0\alpha}(\Gamma)\}$, $\|u\|_{d\alpha,\Gamma} = \max(\|u\|_{T\alpha,\Gamma}, \|\mathrm{Div}\, u\|_{0\alpha,\Gamma})$,

  is the space of Hölder continuous tangential fields on Γ having Hölder continuous surface divergence.

- $X_d^{0\alpha}(\Gamma) = T_d^{0\alpha}(\Gamma)\times C^{0\alpha}(\Gamma)\times C^{0\alpha}(\Gamma)$,

  $\|u\|_{X_d\alpha} = \max(\|u_1\|_{d\alpha,\Gamma}, \|u_2\|_{0\alpha,\Gamma}, \|u_3\|_{0\alpha,\Gamma})$.

- $Y^{0\alpha} = T^{0\alpha}(\Gamma)\times C^{0\alpha}(\Gamma)\times C^{0\alpha}(\Gamma)\times C^{0\alpha}(\Gamma_0)$,

  $\|u\|_{Y\alpha} = \max(\|u_1\|_{T\alpha,\Gamma}, \|u_2\|_{0\alpha,\Gamma}, \|u_3\|_{0\alpha,\Gamma}, \|u_4\|_{0\alpha,\Gamma_0})$.

- $Y_d^{0\alpha} = T_d^{0\alpha}(\Gamma)\times C^{0\alpha}(\Gamma)\times C^{0\alpha}(\Gamma)\times C^{0\alpha}(\Gamma_0)$,

  $\|u\|_{Y_d\alpha} = \max(\|u_1\|_{d\alpha,\Gamma}, \|u_2\|_{0\alpha,\Gamma}, \|u_3\|_{0\alpha,\Gamma}, \|u_4\|_{0\alpha,\Gamma_0})$.

- $\tilde{Y}_d^{0\alpha} = T_d^{0\alpha}(\Gamma)\times C^{0\alpha}(\Gamma)\times C^{1\alpha}(\Gamma)\times C^{0\alpha}(\Gamma_0)$,

  $\|u\|_{\tilde{Y}_d\alpha} = \max(\|u_1\|_{d\alpha,\Gamma}, \|u_2\|_{0\alpha,\Gamma}, \|u_3\|_{1\alpha,\Gamma}, \|u_4\|_{0\alpha,\Gamma_0})$.

Moreover we define

- the bilinear form

  $$\langle\cdot,\cdot\rangle:\ Y^{0\alpha}\times Y^{0\alpha} \to \mathbb{C}, \qquad \langle u,v\rangle = \int_{\Gamma}(u_1\cdot v_1 + u_2 v_2 + u_3 v_3)\, ds + \int_{\Gamma_0} u_4 v_4\, ds,$$

  which is nondegenerated on $Y_d^{0\alpha}\times Y^{0\alpha}$.

- the dual system $(Y_d^{0\alpha}, Y^{0\alpha}; \langle\cdot,\cdot\rangle)$.

- the Hilbert space $L^2_T(\Gamma)$ of L^2 tangential fields on Γ.

- the Hilbert space

  $\mathcal{L}^2 = L^2_T(\Gamma) \times L^2(\Gamma) \times L^2(\Gamma) \times L^2(\Gamma_0)$,

  equipped with the natural scalar product

  $(u, v)_{\mathcal{L}^2} = \langle u, \bar{v} \rangle$.

- the functions Φ and Φ_0 as $\Phi(x,y) = \frac{1}{4\pi} \frac{e^{ik|x-y|}}{|x-y|}$, $\Phi_0(x,y) = \frac{1}{4\pi} \frac{1}{|x-y|}$.

- $[v]$ as the span of v, $[v] = \{x \mid x = zv, \quad z \in \mathbb{C}\}$.

3. Uniqueness

Taking a closer look at (2) we see, that E^L is not uniquely determined, because adding a suitably chosen gradient field does not change the equations.

This is also true for the unbouded problem. Moreover, in the unbounded case we have to prescribe some additional data to make H^L, H^E, E^E unique [8]:

Theorem 1

For the unbounded problem, together with the additional condition

$$\int_{\gamma_i^L} \tau \cdot H^L \, dl = h_i^L, \qquad i = 1,\ldots,p,$$

$h_i^L \in \mathbb{C}$ given, τ being the unit tangent to γ_i^L, the fields H^L, H^E, E^E are uniquely determined.

Problem (2) exhibits the same behaviour, as is shown in the next theorem.

Theorem 2

For the bounded problem (2) together with

$$\int_{\gamma_i^L} \tau \cdot H^L \, dl = h_i^L, \qquad i = 1,\ldots,p, \tag{3}$$

the fields H^L, H^E, E^E are uniquely determined.

Proof

We consider the homogeneous problem with $J_e = 0$ and $h_i^L = 0$, $i = 1,\dots,p$, and show that the fields H^L, H^E, E^E vanish identically.

From the second transmission condition $n \cdot (\mu^L H^L) = n \cdot (\mu^E H^E)$ on Γ, we conclude that for any surface element $S \subset \Gamma$ holds

$$\int_{\partial S} \tau \cdot (E^E - E^L)\, dl = \int_S n \cdot \operatorname{curl}(E^E - E^L)\, ds = i\omega \int_S n \cdot (\mu^E H^E - \mu^L H^L)\, ds = 0,$$

so that the tangential components of $E^E - E^L$ on Γ are of the form

$$(E^E - E^L)|_{\tan} = \operatorname{Grad} \varphi + \sum_{i=1}^{p} e_i^L Z_{0i}^L + \sum_{i=1}^{p} e_i^E Z_i^E,$$

where $\operatorname{Grad} \varphi$ denotes the surface gradient of φ on Γ and $e_i^L, e_i^E \in \mathbb{C}$, $i = 1,\dots,p$, are complex numbers.

In the same way, we derive from

$$\operatorname{curl} H^L = 0 \quad \text{in } G^L, \qquad \int_{\gamma_i^L} \tau \cdot H^L\, dl = 0, \qquad i = 1,\dots,p,$$

that the tangential components of H^L on Γ can be written as a surface gradient

$$H^L|_{\tan} = \operatorname{Grad} \psi.$$

Therefore,

$$\int_\Gamma n \cdot (\overline{H}^L \wedge E^E)\, ds = \int_\Gamma n \cdot (\operatorname{Grad} \overline{\psi} \wedge (E^L + \operatorname{Grad} \varphi + \sum_{i=1}^{p} e_i^L Z_{0i}^L + \sum_{i=1}^{p} e_i^E Z_i^E))\, ds,$$

where $\overline{F}$ denotes the complex conjugate of a field F. By Stokes' theorem we deduce for the terms on the right hand side

$$\int_\Gamma n \cdot (\operatorname{Grad} \overline{\psi} \wedge \operatorname{Grad} \varphi)\, ds = 0,$$

$$\int_\Gamma n \cdot (\operatorname{Grad} \overline{\psi} \wedge Z_{0i}^L)\, ds = 0,$$

$$\int_\Gamma n \cdot (\operatorname{Grad} \overline{\psi} \wedge Z_i^E)\, ds = 0$$

and thus

$$\int_\Gamma n \cdot (\overline{H}^L \wedge E^E)\, ds = \int_\Gamma n \cdot (\overline{H}^L \wedge E^L)\, ds. \tag{4}$$

Using the homogeneous boundary condition $n \cdot (\mu^L H^L) = 0$ on Γ_0 we get for any surface element $S \subset \Gamma_0$

$$\int_{\partial S} \tau \cdot E^L\, dl = \int_S n \cdot \operatorname{rot} E^L\, ds = i\omega \int_S n \cdot (\mu^L H^L)\, ds = 0,$$

so that

$$E^L\big|_{tan} = \operatorname{Grad} \varphi_0 \qquad \text{on } \Gamma_0,$$

since Γ_0 is the boundary of the simply connected domain G_0.

With the same reasoning as above we have

$$H^L\big|_{tan} = \operatorname{Grad} \psi_0 \qquad \text{on } \Gamma_0$$

and a simple application of Stokes' Theorem shows

$$\int_{\Gamma_0} n \cdot (\overline{H}^L \times E^L)\, ds = \int_{\Gamma_0} n \cdot (\operatorname{Grad} \overline{\psi}_0 \times \operatorname{Grad} \varphi_0)\, ds = 0. \tag{5}$$

Using the first transmission condition $n \wedge H^E = n \wedge H^L$ on Γ together with (4) and the Gaussian theorem we get

$$\int_{\Gamma} n \cdot (\overline{H}^L \wedge E^L)\, ds = \int_{\Gamma} n \cdot (\overline{H}^E \wedge E^E)\, ds = \int_{G^E} (\sigma^E E^E \cdot \overline{E}^E - i\omega\mu^E H^E \cdot \overline{H}^E)\, dv.$$

From the application of the Gaussian theorem to the fields in G_0^L follows

$$\begin{aligned}\int_{\Gamma_0} n \cdot (\overline{H}^L \times E^L)\, ds - \int_{\Gamma} n \cdot (\overline{H}^L \times E^L)\, ds &= \int_{\Gamma \cup \Gamma_0} n_0 \cdot (\overline{H}^L \times E^L)\, ds \\ &= \int_{G_0^L} \operatorname{div} (\overline{H}^L \times E^L)\, dv = -\, i\omega\mu^L \int_{G_0^L} H^L \cdot \overline{H}^L\, dv,\end{aligned}$$

where n is the outer normal to G^E resp. G_0 and n_0 is the outer normal to G_0^L. Adding the last two equations, we get with the help of (5)

$$\sigma^E \int_{G^E} E^E \cdot \overline{E}^E\, dv - i\omega \left(\mu^E \int_{G^E} H^E \cdot \overline{H}^E\, dv + \mu^L \int_{G_0^L} H^L \cdot \overline{H}^L\, dv\right) = 0.$$

Since the coefficients ω, μ^L, μ^E and σ^E are positive we finally conlcude

$$H^L \equiv 0, \qquad H^E \equiv 0, \qquad E^E \equiv 0. \qquad \blacksquare$$

4. Existence

For the unbounded problem holds [8]:

Theorem 3

For $J_e \in C^1(\mathbb{R}^3)$, $\operatorname{div} J_e = 0$, $\operatorname{supp}(J_e) \subset G^J$, $\overline{G}^J \subset G^L$ bounded, the unbounded problem (1) is solvable.

In the homogeneous case $J_e = 0$ we get exactly p linear independent solutions H^L, H^E, E^E, where p denotes the topological genus of G^E resp. G^L.

The different solutions are characterized by their circulations

$$\int_{\gamma_i^L} \tau \cdot H^L \, dl = h_i^L, \qquad i = 1,\dots,p,$$

along γ_i^L.

E^L is not uniquely determined.

To show existence for (2) we consider the following auxiliary problem

Find $H^L \in C^1(G_0^L) \cap C(\overline{G}_0^L)$,

$$H^E \in C^2(G^E) \cap C(\overline{G}^E), \quad \operatorname{div} H^E \in C(\overline{G}^E), \quad \operatorname{curl} H^E \in C(\overline{G}^E)$$

solving

$$\begin{array}{lclc} \operatorname{curl} H^L = 0 & \text{in } G_0^L, & (\Delta + k^2) H^E = 0 & \text{in } G^E, \\ \operatorname{div} H^L = 0 & & k^2 = i\omega\sigma^E\mu^E & \end{array}$$

$$\begin{array}{ll} n \wedge H^E - n \wedge H^L = c & \\ n \cdot (\mu^E H^E) - n \cdot (\mu^L H^L) = g & \text{on } \Gamma, \\ \operatorname{div} H^E = d & \\ n \cdot (\mu^L H^L) = f & \text{on } \Gamma_0, \\ \int_{\gamma_i^L} \tau \cdot H^L \, dl = 0 & i = 1,\dots,p, \end{array} \tag{6}$$

where we choose k so that $\operatorname{Im}(k) > 0$.

We show that (6) is uniquely solvable for sufficiently smooth data c,g,d,f. Using the solutions of (6), we solve our original problem (2). In this process, the following Lemma, which is shown in [9], plays an important role.

Lemma 1

Let $H \in C^1(G_0^L) \cap C(\overline{G}_0^L)$ satisfy $\operatorname{div} H = 0$ in G_0^L and

$$\int_{\Gamma_j} n \cdot H \, ds = 0, \qquad j = 1,\dots,m$$

for any connected component Γ_j of Γ. There exists a field Field $E \in C^1(G_0^L) \cap C(\overline{G}_0^L)$ such that

$$\operatorname{curl} E = i\omega\mu^L H, \qquad \operatorname{div} E = 0 \qquad \text{in } G_0^L.$$

With the help of this Lemma, we prove uniqueness for (6).

Lemma 2

Problem (6) has at most one solution.

Proof

Let H^L, H^E be solutions of the homogeneous problem. The divergence of H^E vanishes identically in G^E, because $\operatorname{div} H^E\big|_\Gamma = 0$, $\operatorname{Im}(k) > 0$ and $\operatorname{div} H^E$ is a solution of the scalar Helmholtz equation in G^E with wave number k [1]. Therefore, H^E and $E^E = \frac{1}{\sigma^E} \operatorname{curl} H^E$ solve the Maxwell equations in G^E. Moreover from $g = 0$ we get

$$\mu^L \int_{\Gamma_j} n \cdot H^L \, ds = \int_{\Gamma_j} n \cdot (\mu^L H^L) \, ds = \int_{\Gamma_j} n \cdot (\mu^E H^E) \, ds$$
$$= \frac{1}{i\omega} \int_{\Gamma_j} n \cdot (i\omega\mu^E H^E) \, ds = \frac{1}{i\omega} \int_{\Gamma_j} n \cdot \operatorname{curl} E^E \, ds = 0,$$

so that Lemma 1 can be applied to H^L. Lemma 1 guarantees the existence of E^L,

$$\operatorname{curl} E^L = i\omega\mu^L H^L \quad \text{in } G_0^L.$$

But now, the fields H^L, E^L, H^E, E^E are solutions of (2) with homogeneous circulations (3). Using Theorem 2, we get

$$H^L \equiv 0, \; H^E \equiv 0. \qquad \blacksquare$$

For the solution of the auxiliary problem (6), we make the following ansatz :

Lemma 3

Define H^L and H^E as

$$H^L(x) = \operatorname{grad}_x \int_\Gamma \lambda(y)\, \Phi_0(x,y)\, ds(y) + \operatorname{grad}_x \int_{\Gamma_0} \varkappa(y)\, \Phi_0(x,y)\, ds(y),$$

$$H^E(x) = \operatorname{curl}_x \int_\Gamma a(y)\, \Phi(x,y)\, ds(y) + \operatorname{grad}_x \int_\Gamma \lambda(y)\, \Phi(x,y)\, ds(y)$$
$$+ \int_\Gamma n(y)\, \delta(y)\, \Phi(x,y)\, ds(y),$$

$$a \in T_d^{0\alpha}(\Gamma), \quad \lambda \in C^{0\alpha}(\Gamma), \quad \delta \in C^{0\alpha}(\Gamma), \quad \varkappa \in C^{0\alpha}(\Gamma_0).$$

Then

$$H^L \in C^\infty(G_0^L) \cap C^{0\alpha}(\overline{G}_0^L),$$
$$H^E \in C^2(G^E) \cap C^{0\alpha}(\overline{G}^E), \quad \operatorname{div} H^E \in C^{0\alpha}(\overline{G}^E), \quad \operatorname{curl} H^E \in C^{0\alpha}(\overline{G}^E),$$

$$\left.\begin{array}{l}\operatorname{curl} H^L = 0\\ \operatorname{div} H^L = 0\end{array}\right\} \text{ in } G_0^L, \qquad \left.\begin{array}{l}(\Delta + k^2)\, H^E = 0\\ k^2 = i\omega\sigma^E\mu^E\end{array}\right\} \text{ in } G^E,$$

$$\int_{\gamma_i^L} \tau \cdot H^L \, dl = 0 \qquad i = 1,\dots,p.$$

Proof

The regularity properties of H^L and H^E follow from corresponding theorems about single and double layer potentials in [1]. H^L and H^E obviously solve the required differential equations. Since H^L is a gradient field, the circulations along γ_i^L vanish. ■

For the values of H^L, H^E on the boundaries follows

Lemma 4

Defining $F_\pm$ as $F_\pm(x) = \lim_{h \searrow 0} F(x \pm h\, n(x))$ for $x \in \Gamma$ or $x \in \Gamma_0$, we get

$$H_+^L(x) = \int_\Gamma \lambda(y) \operatorname{grad}_x \Phi_0(x,y)\, ds(y) - \frac{1}{2} n(x)\, \lambda(x) + \int_{\Gamma_0} \varkappa(y) \operatorname{grad}_x \Phi_0(x,y)\, ds(y),$$

$$\begin{aligned} H_-^E(x) = &\int_\Gamma \operatorname{curl}_x(a(y)\, \Phi(x,y))\, ds(y) + \frac{1}{2} n(x) \wedge a(x) \\ &+ \int_\Gamma \lambda(y) \operatorname{grad}_x \Phi(x,y)\, ds(y) + \frac{1}{2} n(x)\, \lambda(x) \\ &+ \int_\Gamma n(y)\, \delta(y)\, \Phi(x,y)\, ds(y), \end{aligned}$$

$$\begin{aligned} (\operatorname{div} H^E)_-(x) = &- k^2 \int_\Gamma \lambda(y)\, \Phi(x,y)\, ds(y) \\ &- \int_\Gamma \delta(y)\, \partial_{n_y} \Phi(x,y)\, ds(y) + \frac{1}{2} \delta(x) \end{aligned}$$

on Γ resp.

$$H^L_-(x) = \int_\Gamma \lambda(y)\, \mathrm{grad}_x \Phi_0(x,y)\, ds(y) + \int_{\Gamma_0} \varkappa(y)\, \mathrm{grad}_x \Phi_0(x,y)\, ds(y) + \frac{1}{2} n(x)\, \varkappa(x)$$

on Γ_0, where n is the outer normal to G^E resp. G_0.

Proof

The jump conditions for single and double layer potentials and their derivatives are given by [1]:

$$\begin{aligned}
\mathrm{curl}_x \int_\Gamma a(y)\, \Phi(x,y)\, ds(y)\Big|_\pm &= \int_\Gamma \mathrm{curl}_x(a(y)\, \Phi(x,y))\, ds(y) \mp \frac{1}{2} n(x) \wedge a(x), \\
\mathrm{grad}_x \int_\Gamma \lambda(y)\, \Phi(x,y)\, ds(y)\Big|_\pm &= \int_\Gamma \lambda(y)\, \mathrm{grad}_x \Phi(x,y)\, ds(y) \mp \frac{1}{2} n(x)\, \lambda(x), \\
\int_\Gamma n(y)\, \delta(y)\, \Phi(x,y)\, ds(y)\Big|_\pm &= \int_\Gamma n(y)\, \delta(y)\, \Phi(x,y)\, ds(y), \\
\int_\Gamma \lambda(y)\, \Phi(x,y)\, ds(y)\Big|_\pm &= \int_\Gamma \lambda(y)\, \Phi(x,y)\, ds(y), \\
\int_\Gamma \delta(y)\, \partial_{n_y}\Phi(x,y)\, ds(y)\Big|_\pm &= \int_\Gamma \delta(y)\, \partial_{n_y}\Phi(x,y)\, ds(y) \pm \frac{1}{2} \delta(x).
\end{aligned} \tag{7}$$

They do not change if we replace Φ by Φ_0 or Γ by Γ_0.

Thus we immediately get the representation of H^L_+, H^E_- on Γ and H^L_- on Γ_0. For $(\mathrm{div}\, H^E)_-$ on Γ we use

$$\begin{aligned}
(\mathrm{div}\, H^E)(x) &= \Delta \int_\Gamma \lambda(y)\, \Phi(x,y)\, ds(y) + \int_\Gamma \mathrm{div}_x(n(y)\, \delta(y)\, \Phi(x,y))\, ds(y) \\
&= -k^2 \int_\Gamma \lambda(y)\, \Phi(x,y)\, ds(y) + \int_\Gamma \delta(y)\, n(y) \cdot \mathrm{grad}_x \Phi(x,y)\, ds(y) \\
&= -k^2 \int_\Gamma \lambda(y)\, \Phi(x,y)\, ds(y) - \int_\Gamma \delta(y)\, \partial_{n_y}\Phi(x,y)\, ds(y). \qquad \blacksquare
\end{aligned}$$

Thus our ansatz solves the differential equations of (6), so that only the boundary conditions on Γ resp. Γ_0 have to be adjusted. This leads to a boundary integral equation on $\Gamma \cup \Gamma_0$.

In the subsequent Lemmata we make use of the following operators

Definition

$$(Ma)(x) = 2\, n(x) \wedge \int_\Gamma \mathrm{curl}_x(a(y)\, \Phi(x,y))\, ds(y), \qquad x \in \Gamma,$$

$$(Na)(x) = 2\, n(x) \cdot \int_\Gamma \mathrm{curl}_x(a(y)\, \Phi(x,y))\, ds(y), \qquad x \in \Gamma,$$

$$(K\lambda)(x) = 2\int_\Gamma \lambda(y)\, \partial_{n_y}\Phi(x,y)\, ds(y), \qquad x \in \Gamma,$$

$$(K'\lambda)(x) = 2\int_\Gamma \lambda(y)\, \partial_{n_x}\Phi(x,y)\, ds(y), \qquad x \in \Gamma,$$

$$(S\lambda)(x) = 2\int_\Gamma \lambda(y)\, \Phi(x,y)\, ds(y), \qquad x \in \Gamma,$$

$$(P\lambda)(x) = 2\, n(x) \wedge \int_\Gamma n(y)\, \lambda(y)\, \Phi(x,y)\, ds(y), \qquad x \in \Gamma,$$

$$(Q\lambda)(x) = 2\, n(x) \cdot \int_\Gamma n(y)\, \lambda(y)\, \Phi(x,y)\, ds(y), \qquad x \in \Gamma,$$

$$(R\lambda)(x) = 2\, n(x) \wedge \int_\Gamma \lambda(y)\, \mathrm{grad}_x \Phi(x,y)\, ds(y), \qquad x \in \Gamma,$$

$$(K'^{0}\varkappa)(x) = 2\int_{\Gamma_0} \varkappa(y)\, \partial_{n_x}\Phi(x,y)\, ds(y), \qquad x \in \Gamma_0,$$

$$(K'^{\Gamma_0\Gamma}\varkappa)(x) = 2\int_{\Gamma_0} \varkappa(y)\, \partial_{n_x}\Phi(x,y)\, ds(y), \qquad x \in \Gamma,$$

$$(K'^{\Gamma\Gamma_0}\lambda)(x) = 2\int_{\Gamma} \lambda(y)\, \partial_{n_x}\Phi(x,y)\, ds(y), \qquad x \in \Gamma_0,$$

$$(R^{\Gamma_0\Gamma}\varkappa)(x) = 2\, n(x) \wedge \int_{\Gamma_0} \varkappa(y)\, \mathrm{grad}_x \Phi(x,y)\, ds(y), \qquad x \in \Gamma.$$

If Φ is replaced by Φ_0, the operators are marked with a lower index $_0$.

Lemma 5

The integral operators defined above have the following mapping properties

$M: T^{0\alpha}(\Gamma) \to T^{0\alpha}(\Gamma)$ resp. $T_d^{0\alpha}(\Gamma) \to T_d^{0\alpha}(\Gamma)$, $\quad N: T_d^{0\alpha}(\Gamma) \to C^{0\alpha}(\Gamma)$,

$K, K_0, K', K'_0, S, Q: C^{0\alpha}(\Gamma) \to C^{0\alpha}(\Gamma)$, $\quad P: C^{0\alpha}(\Gamma) \to T_d^{0\alpha}(\Gamma)$,

$R, R_0: C^{0\alpha}(\Gamma) \to T_d^{0\alpha}(\Gamma)$, $\quad K_0'^{0}: C^{0\alpha}(\Gamma_0) \to C^{0\alpha}(\Gamma_0)$,

$K_0'^{\Gamma_0\Gamma}: C^{0\alpha}(\Gamma_0) \to C^{0\alpha}(\Gamma)$, $\quad K_0'^{\Gamma\Gamma_0}: C^{0\alpha}(\Gamma) \to C^{0\alpha}(\Gamma_0)$,

$R_0^{\Gamma_0\Gamma}: C^{0\alpha}(\Gamma_0) \to T_d^{0\alpha}(\Gamma)$.

N, R, R_0 are continuous, $M, K, K_0, K', K'_0, S, Q, P, R - R_0, K_0'^{0}, K_0'^{\Gamma_0\Gamma}, K_0'^{\Gamma\Gamma_0}, R_0^{\Gamma_0\Gamma}$ are compact.

Proof

In [1,2,3,10] the continuity of N, the compactness of $M, K, K_0, K', K'_0, S, Q, P, R - R_0, K_0'^{0}$ are shown as well as the continuity of R, R_0 resp. compactness of

$R - R_0$ where R, R_0 are regarded as operators mapping $C^{0\alpha}(\Gamma_0)$ to $T^{0\alpha}(\Gamma)$.

Setting

$$F(x) = 2\int_\Gamma \lambda(y)\, \mathrm{grad}_x \Phi(x,y)\, ds(y), \qquad \lambda \in C^{0\alpha}(\Gamma), \qquad x \in G^E,$$

we get

$$F \in C^2(G^E) \cap C^{0\alpha}(\bar{G}^E),$$

$$\mathrm{curl}\, F = 0 \qquad \text{in } G^E,$$

$$n \wedge F\big|_\Gamma = R\lambda.$$

According to [1] we deduce

$$\mathrm{Div}\,(R\lambda) = \mathrm{Div}\,(n \wedge F) = -n \cdot \mathrm{curl}\, F\big|_\Gamma = 0$$

and therefore

$$\|R\lambda\|_{d\alpha,\Gamma} = \|R\lambda\|_{T\alpha,\Gamma}.$$

In the same way we show $\|R_0\lambda\|_{d\alpha,\Gamma} = \|R_0\lambda\|_{T\alpha,\Gamma}$, so that R, R_0 and $R - R_0$ have the same properties as before if we replace $T^{0\alpha}(\Gamma)$ by $T_d^{0\alpha}(\Gamma)$.

The compactness of $K_0'^{\Gamma_0\Gamma}, K_0'^{\Gamma\Gamma_0}, R_0^{\Gamma_0\Gamma}$ is obvious. ■

Lemma 6

H^L, H^E defined in Lemma 3, solve (6), if $a, \lambda, \delta, \varkappa$ are a solution of the integral equation $Av = b$,

$$A = \begin{pmatrix} M - I & R - R_0 & P & -R_0^{\Gamma_0\Gamma} \\ \mu^E N & \mu^E(I + K') + \mu^L(I - K_0') & \mu^E Q & -\mu^L K_0'^{\Gamma_0\Gamma} \\ 0 & -k^2 S & I - K & 0 \\ 0 & \mu^L K_0'^{\Gamma\Gamma_0} & 0 & \mu^L(I + K_0'^0) \end{pmatrix},$$

$$v = \begin{pmatrix} a \\ \lambda \\ \delta \\ \varkappa \end{pmatrix}, \qquad b = 2\begin{pmatrix} c \\ g \\ d \\ f \end{pmatrix}.$$

Proof

This is obvious from the definition of the operators and the representation of the values of $H^L, H^E, \mathrm{div}\, H^E$ on the boundaries given in Lemma 4. ■

To solve the above integral equation, we want to apply Fredholm theory.

Lemma 7

The operator A can be decomposed into $A = B + C$,

$$B = \begin{pmatrix} -I & 0 & 0 & 0 \\ \mu^E N & (\mu^E + \mu^L) I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & \mu^L I \end{pmatrix},$$

$$C = \begin{pmatrix} M & R - R_0 & P & -R_0^{\Gamma_0 \Gamma} \\ 0 & \mu^E K' - \mu^L K_0' & \mu^E Q & -\mu^L K_0'^{\Gamma_0 \Gamma} \\ 0 & -k^2 S & -K & 0 \\ 0 & \mu^L K_0'^{\Gamma \Gamma_0} & 0 & \mu^L K_0'^{0} \end{pmatrix}.$$

B, C map $Y_d^{0\alpha}$ into itself. B is continuously invertible, C is compact.
The null space N(A) has dimension 1.

Proof

The mapping properties of B and C follow directly from Lemma 5. To determine the nullspace of A, we proceed in two steps. We first show, that $\dim N(A) \le 1$ and then $\dim N(A) \ne 0$.

(i) $\dim N(A) \le 1$

Let v be a solution of the homogeneous equation $Av = 0$. Using v to define H^L and H^E according to Lemma 3, we know from Lemma 6, that H^L, H^E solve the homogeneous auxiliary problem (6). By Lemma 2, this problem possesses at most one solution and therefore $H^L \equiv 0$ in G_0^L. But H^L was defined as

$$H^L(x) = \operatorname{grad}_x \int_\Gamma \lambda(y)\, \Phi_0(x,y)\, ds(y) + \operatorname{grad}_x \int_{\Gamma_0} \varkappa(y)\, \Phi_0(x,y)\, ds(y),$$

$$= \operatorname{grad}_x \int_{\Gamma \cup \Gamma_0} \tilde{\lambda}(y)\, \Phi_0(x,y)\, ds(y), \qquad \tilde{\lambda}(x) = \begin{cases} \lambda(x), & x \in \Gamma \\ \varkappa(x), & x \in \Gamma_0 \end{cases} \in C^{0\alpha}(\Gamma \cup \Gamma_0)$$

Applying the jump conditions (7) to H^L on $\Gamma \cup \Gamma_0 = \partial G_0^L$ we get

$$0 = 2 H_-^L = 2 \int_{\Gamma \cup \Gamma_0} \tilde{\lambda}(y)\, \partial_{n_x} \Phi_0(x,y)\, ds(y) + \tilde{\lambda}(x) = (I + \tilde{K}_0')\, \tilde{\lambda} \qquad \text{on } \Gamma \cup \Gamma_0,$$

where n denotes the outer normal to G_0^L and $\tilde{K}_0' : C^{0\alpha}(\Gamma \cup \Gamma_0) \to C^{0\alpha}(\Gamma \cup \Gamma_0)$ is defined in analogy to K_0' on $C^{0\alpha}(\Gamma)$. But this is the integral equation for the solution of the interior harmonic Neumann boundary-value problem in G_0^L with the help of a single layer potential ansatz. Since G_0^L is a bounded, connected domain, we have [4]

$$N(I + K_0') = [\,\tilde{\psi}\,] := \{\eta \mid \eta = z\tilde{\psi},\ z \in \mathbb{C}\}, \qquad \tilde{\psi} \in C^{0\alpha}(\Gamma \cup \Gamma_0)$$

and

$$\varkappa = z\,\tilde{\psi}\big|_{\Gamma_0},$$

for some $z \in \mathbb{C}$. Substituting this result into the homogeneous integral equation $Av = 0$, we get

$$\tilde{A}\begin{pmatrix} a \\ \lambda \\ \delta \end{pmatrix} = \begin{pmatrix} R_0^{\Gamma_0\Gamma}\varkappa \\ \mu^L K_0'^{\Gamma_0\Gamma}\varkappa \\ 0 \end{pmatrix} \in X_d^{0\alpha}, \qquad \tilde{A} = \begin{pmatrix} M - I & R - R_0 & P \\ \mu^E N & \mu^E(I + K') + \mu^L(I - K_0') & \mu^E Q \\ 0 & -k^2 S & I - K \end{pmatrix}.$$

But according to [8], $\tilde{A}$ is continuously invertible in $X_d^{0\alpha}(\Gamma)$. Therefore a, λ, δ are uniquely determined by $\varkappa$ so that $\dim N(A) \le 1$.

(ii) $\dim N(A) > 0$.

Now suppose that $\dim N(A) = 0$. In this case, A would be continuously invertible in $Y_d^{0\alpha}$ and the auxiliary problem (6) would be solvable for any choice of $(c, g, d, f)^T \in Y_d^{0\alpha}$. We consider $(0,0,0,f)^T$, $f \in C^{0\alpha}(\Gamma_0)$ arbitrary, and get

$$\begin{aligned} 0 &= \mu^L \int_{G_0^L} \operatorname{div} H^L\, dv = \int_{\Gamma_0} n\cdot(\mu^L H^L)\, ds - \int_{\Gamma} n\cdot(\mu^L H^L)\, ds \\ &= \int_{\Gamma_0} f\, ds - \int_{\Gamma} n\cdot(\mu^E H^E)\, ds = \int_{\Gamma_0} f\, ds - \mu^E \int_{G^E} \operatorname{div} H^E\, dv, \end{aligned}$$

where n is the outer normal to G_0 resp. G^L. Since $d = 0$, we know that $\operatorname{div} H^E = 0$ in G^E and thus

$$\int_{\Gamma_0} f\, ds = 0 \qquad \forall f \in C^{0\alpha}(\Gamma_0),$$

which is of course not true. Therefore $\dim N(A) > 0$. ∎

Since A is noninjective, we have to determine the nullspace of the adjoint operator A'. Before we define A', we introduce some new notations.

Definition

$$(M'b)(x) = n(x) \wedge (M(n \wedge b))(x), \qquad x \in \Gamma,$$

$$(N'\lambda)(x) = -2\, n(x) \wedge (n(x) \wedge \int_\Gamma \mathrm{curl}_x (n(y)\,\lambda(y)\,\Phi(x,y))\, ds(y)), \qquad x \in \Gamma,$$

$$(P'a)(x) = -2\, n(x) \cdot \int_\Gamma n(y) \wedge a(y)\, \Phi(x,y)\, ds(y), \qquad x \in \Gamma,$$

$$(R'a)(x) = 2 \int_\Gamma \mathrm{div}_x (n(y) \wedge a(y)\, \Phi(x,y))\, ds(y), \qquad x \in \Gamma,$$

$$(K^0 \varkappa)(x) = 2 \int_{\Gamma_0} \varkappa(y)\, \partial_{n_y} \Phi(x,y)\, ds(y), \qquad x \in \Gamma_0,$$

$$(K^{\Gamma_0\Gamma} \varkappa)(x) = 2 \int_{\Gamma_0} \varkappa(y)\, \partial_{n_y} \Phi(x,y)\, ds(y), \qquad x \in \Gamma,$$

$$(K^{\Gamma\Gamma_0} \lambda)(x) = 2 \int_\Gamma \lambda(y)\, \partial_{n_y} \Phi(x,y)\, ds(y), \qquad x \in \Gamma_0,$$

$$(R'^{\Gamma\Gamma_0} a)(x) = 2 \int_\Gamma \mathrm{div}_x (n(y) \wedge a(y)\, \Phi(x,y))\, ds(y), \qquad x \in \Gamma_0.$$

We use again the subscript $_0$ to indicate that Φ is replaced by Φ_0.

Lemma 8

For the operaters defined above we get

$$M' : T^{0\alpha}(\Gamma) \to T^{0\alpha}(\Gamma), \qquad N' : C^{0\alpha}(\Gamma) \to T^{0\alpha}(\Gamma),$$

$$P' : T^{0\alpha}(\Gamma) \to C^{0\alpha}(\Gamma), \qquad R', R'_0 : T^{0\alpha}(\Gamma) \to C^{0\alpha}(\Gamma),$$

$$K_0^0 : C^{0\alpha}(\Gamma_0) \to C^{0\alpha}(\Gamma_0),$$

$$K_0^{\Gamma_0\Gamma} : C^{0\alpha}(\Gamma_0) \to C^{0\alpha}(\Gamma), \qquad K_0^{\Gamma\Gamma_0} : C^{0\alpha}(\Gamma) \to C^{0\alpha}(\Gamma_0),$$

$$R_0'^{\Gamma\Gamma_0} : T^{0\alpha}(\Gamma) \to C^{0\alpha}(\Gamma_0).$$

N', R', R'_0 are continuous, $M', P', R' - R'_0, K_0^0, K_0^{\Gamma_0\Gamma}, K_0^{\Gamma\Gamma_0}, R_0'^{\Gamma\Gamma_0}$ are compact.

Moreover the operators B', C', defined by

$$B' = \begin{pmatrix} -I & \mu^E N' & 0 & 0 \\ 0 & (\mu^L + \mu^E) I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & \mu^L I \end{pmatrix}, \qquad C' = \begin{pmatrix} M' & 0 & 0 & 0 \\ R' - R'_0 & \mu^E K - \mu^L K_0 & -k^2 S & \mu^L K_0^{\Gamma_0\Gamma} \\ P' & \mu^E Q & -K' & 0 \\ -R_0'^{\Gamma\Gamma_0} & -\mu^L K_0^{\Gamma\Gamma_0} & 0 & \mu^L K_0^0 \end{pmatrix}$$

map $Y^{0\alpha}$ into itself. B' is continuously invertible, C' is compact.

The operator

$$A' = B' + C' \;:\; Y^{0\alpha} \to Y^{0\alpha},$$

is the adjoint of A with respect to the dual system $(Y_d^{0\alpha}, Y^{0\alpha}, \langle\cdot,\cdot\rangle)$.

Proof

The mapping properties of the operators $N', R', R'_0, M', P', R' - R'_0$ are shown in [1,3,10]. The results for $K_0^0, K_0^{\Gamma_0\Gamma}, K_0^{\Gamma\Gamma_0}, R_0'^{\Gamma\Gamma_0}$ are obvious. Together with Lemma 5 we immediately get the invertibility of B' and the compactness of C'.

That A' is the adjoint of A is shown by simple but lengthy calculations.

■

Lemma 9

Define $\mathcal{N}(\Gamma)$ as

$$\mathcal{N}(\Gamma) = \{\psi \mid \psi \in C(\Gamma),\; K\psi \text{ possesses continuous normal derivatives on both sides of } \Gamma\}.$$

Then

$$T: \mathcal{N}(\Gamma) \to C(\Gamma)$$

$$\psi \mapsto 2\,\partial_{n_x} \int_\Gamma \psi(y)\,\partial_{n_y}\Phi(x,y)\,ds(y)$$

is well defined. Moreover $C^{1\alpha}(\Gamma) \subset \mathcal{N}(\Gamma)$, $T: C^{1\alpha}(\Gamma) \to C^{0\alpha}(\Gamma)$ is continuous. $\frac{1}{2}T: \mathcal{N}(\Gamma) \to C(\Gamma)$ maps the density of a double layer potential to its normal derivative and

$$\langle T\varphi, \psi\rangle_\Gamma = \langle \varphi, T\psi\rangle_\Gamma \qquad \forall\, \varphi,\psi \in \mathcal{N}(\Gamma),$$

with $\langle u, v\rangle_\Gamma = \int_\Gamma u(y)\,v(y)\,ds(y)$.

Proof

See [1].

■

Lemma 10

$$N(A') = [(0,1,d',1)^T], \qquad d' = -\frac{\mu}{k^2}(I - K')^{-1}T1.$$

Proof

From Lemma 7, Lemma 8 and the Fredholm alternative we know, that $\dim N(A') = 1$. Therefore, $N(A') = [b']$, $b' \in Y^{0\alpha}$.

Next we define the subspaces $U = [\bar{b}']^{\perp L^2}$ and $W = [\bar{b}']^{\perp L^2} \cap \tilde{Y}_d^{0\alpha}$ of L^2 and consider an element $b = (c, g, d, f)^T \in W$. From the definition of W we know that

$$<b, b'> = (b, \bar{b}')_{L^2} = 0,$$

so that $Av = b$ and therefore the auxiliary problem (6) is solvable. For the corresponding solution of (6) we get

$$\begin{aligned} 0 = \mu^L \int_{G_0^L} \operatorname{div} H^L \, dv &= \int_{\Gamma_0} n \cdot (\mu^L H^L) \, ds - \int_{\Gamma} n \cdot (\mu^L H^L) \, ds \\ &= \int_{\Gamma_0} f \, ds - \int_{\Gamma} (n \cdot (\mu^E H^E) - g) \, ds \\ &= \int_{\Gamma_0} f \, ds + \int_{\Gamma} g \, ds - \mu^E \int_{G^E} \operatorname{div} H^E \, dv. \end{aligned} \tag{8}$$

$D^E = \operatorname{div} H^E$ solves the scalar Helmholtz equation in G^E with Dirichlet boundary-value d. Since $b \in \tilde{Y}_d^{0\alpha}$ we have $d \in C^{1\alpha}(\Gamma)$ and $D^E \in C^{1\alpha}(\bar{G}^E)$ [1]. Thus we deduce from (8)

$$\begin{aligned} 0 &= \int_{\Gamma_0} f \, ds + \int_{\Gamma} g \, ds - \mu^E \int_{G^E} \operatorname{div} H^E \, dv = \int_{\Gamma_0} f \, ds + \int_{\Gamma} g \, ds + \frac{\mu^E}{k^2} \int_{G^E} \Delta D^E \, dv \\ &= \int_{\Gamma_0} f \, ds + \int_{\Gamma} g \, ds + \frac{\mu}{k^2} \int_{\Gamma} \partial_n D^E \, ds. \end{aligned}$$

But according to [1], D^E may be represented as a double layer potential

$$D^E = \int_{\Gamma} \psi(y) \, \partial_{n_y} \Phi(x,y) \, ds(y) \qquad \text{in } G^E,$$

$$\psi = -2 (I - K)^{-1} d \in C^{1\alpha}(\Gamma).$$

Since $\psi \in C^{1\alpha}(\Gamma) \subset N(\Gamma)$, we get with the help of Lemma 9

$$\partial_n D^E \big|_{\Gamma} = \frac{1}{2} T\psi$$

so that

$$0 = \int_{\Gamma_0} f \, ds + \int_{\Gamma} g \, ds + \frac{\mu^E}{2k^2} \int_{\Gamma} T\psi \, ds.$$

Using again Lemma 9 we conclude

$$\begin{aligned} \int_{\Gamma} T\psi \, ds &= <T\psi, 1>_{\Gamma} = <\psi, T1>_{\Gamma} = -2 <(I-K)^{-1} d, T1>_{\Gamma} \\ &= -2 <d, (I-K')^{-1} T1>_{\Gamma} = -2 \int_{\Gamma} d \, ((I-K')^{-1} T1) \, ds \end{aligned}$$

since K' is the adjoint of K with respect to the dual system $(C^{0\alpha}(\Gamma), C^{0\alpha}(\Gamma), <\cdot,\cdot>_\Gamma)$.
So we have

$$0 = \int_{\Gamma_0} f\, ds + \int_{\Gamma} g\, ds - \frac{\mu}{k^2}\int_{\Gamma} d((I-K')^{-1}T1)\, ds = < \begin{pmatrix} c \\ g \\ d \\ f \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ d' \\ 1 \end{pmatrix} > = (\begin{pmatrix} c \\ g \\ d \\ f \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ d' \\ 1 \end{pmatrix})_{L^2}$$

$$\Rightarrow \quad (b, \begin{pmatrix} 0 \\ 1 \\ d' \\ 1 \end{pmatrix})_{L^2} = 0, \qquad \forall\, b \in W = U \cap \tilde{Y}_d^{0\alpha} = [\,\bar{b}'\,]^{\perp L^2} \cap \tilde{Y}_d^{0\alpha}$$

It is easily shown, that W is dense in $U = [\,\bar{b}'\,]^{\perp L^2}$. Therefore

$$(b, \begin{pmatrix} 0 \\ 1 \\ d' \\ 1 \end{pmatrix})_{L^2} = 0 \qquad \forall\, b \in U.$$

$$\Rightarrow \begin{pmatrix} 0 \\ 1 \\ d' \\ 1 \end{pmatrix} \in U^{\perp L^2} = ([\,\bar{b}'\,]^{\perp L^2})^{\perp L^2} = [\,\bar{b}'\,].$$

$$\Rightarrow N(A') = [\,b'\,] = [\begin{pmatrix} 0 \\ 1 \\ d' \\ 1 \end{pmatrix}].$$

■

A direct consequence of the last Lemma is the following theorem.

Theorem 4

The auxiliary problem (6) is uniquely solvable, if

$$\begin{pmatrix} c \\ g \\ d \\ f \end{pmatrix} \in Y_d^{0\alpha}, \qquad < \begin{pmatrix} c \\ g \\ d \\ f \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ d' \\ 1 \end{pmatrix} > = 0, \qquad d' = -\frac{\mu^E}{k^2}(I-K')^{-1}T1.$$

Proof

Follows from the Lemmata 6, 7, 8, 10, the Fredholm alternative and Lemma 2.

■

Lemma 11

Consider $J_e \in C^1(\mathbb{R}^3)$, $\operatorname{div} J_e = 0$, $\operatorname{supp}(J_e) \subset G^J$. There exists a vectorfield $H^J \in C^1(\mathbb{R}^3)$ with

$$\operatorname{curl} H^J = J_e, \quad \operatorname{div} H^J = 0.$$

Proof

$J_e \in C^1(\mathbb{R}^3)$ has compact support, so that $J_e \in C^{0\alpha}(\mathbb{R}^3)$. Using some well known regularity results for Newtonian potentials [5], we get

$$A = \int_{G^J} J_e(y)\, \Phi_0(x,y)\, dy \quad \in \quad C^2(\mathbb{R}^3).$$

and defining H^J as $H^J = \operatorname{curl} A \in C^1(\mathbb{R}^3)$ we see that

$$\operatorname{div} H^J = 0, \qquad \operatorname{curl} H^J = \operatorname{curl}\operatorname{curl} A = (\operatorname{grad}\operatorname{div} - \Delta)\, A = J_e. \qquad \blacksquare$$

Using Theorem 4 and Lemma 11, we are able to prove the main result of this paper.

Theorem 5

Consider $J_e \in C^1(\mathbb{R}^3)$, $\operatorname{div} J_e = 0$, $\operatorname{supp}(J_e) \subset G^J$, $\overline{G}^J \subset G_0^L$, G^J bounded, $f \in C^{0\alpha}(\Gamma_0)$. The bounded problem (2) together with (3) is solvable if and only if

$$\int_{\Gamma_0} f\, ds = 0.$$

H^L, H^E, E^E are uniquely determined.

Proof

Suppose (2), (3) is solvable. Then

$$\begin{aligned}
\int_{\Gamma_0} f\, ds &= \int_{\Gamma_0} n \cdot (\mu^L H^L)\, ds \\
&= \int_{\Gamma_0} n \cdot (\mu^L H^L)\, ds - \int_{\Gamma} n \cdot (\mu^L H^L)\, ds + \int_{\Gamma} n \cdot (\mu^L H^L)\, ds \\
&= \mu^L \int_{G_0^L} \operatorname{div} H^L\, dv + \int_{\Gamma} n \cdot (\mu^E H^E)\, ds \\
&= \frac{1}{i\omega} \int_{\Gamma} n \cdot \operatorname{curl} E^E\, ds = 0
\end{aligned}$$

by Stokes' Theorem.

For the "if" part, we consider H^J, which is given by Lemma 11, and define

$$h_i^J = \int_{\gamma_i^L} \tau \cdot H^J\, dl, \qquad i = 1,\ldots,p,$$

$$H^Z = \sum_{j=1}^{p} (h_j^L - h_j^J)\, Z_{0j}^L,$$

where h_i^L are the given circulations from (3). From the regularity properties of H^J and Z_{0j}^L, we get

$$c = n \wedge (H^J + H^Z)\big|_\Gamma \in T^{0\alpha}(\Gamma),$$
$$g = n \cdot (\mu^L (H^J + H^Z))\big|_\Gamma = n \cdot (\mu^L H^J)\big|_\Gamma \in C^{0\alpha}(\Gamma),$$
$$\tilde{f} = f - n \cdot (\mu^L H^J)\big|_{\Gamma_0} \in C^{0\alpha}(\Gamma_0).$$

Moreover, for the surface divergence of c on Γ holds

$$\mathrm{Div}\, c = \mathrm{Div}(n \wedge (H^J + H^Z)\big|_\Gamma) = -\, n \cdot \mathrm{curl}(H^J + H^Z)\big|_\Gamma = -\, n \cdot J_e\big|_\Gamma = 0,$$

so that $c \in T_d^{0\alpha}(\Gamma)$.

From the proof of Lemma 11 we know, that H^J may be represented by

$$H^J = \mathrm{curl}\, A, \qquad A \in C^2(\mathbb{R}^3).$$

Therefore, using Stokes' theorem, we obtain

$$\int_{\Gamma_j} g\, ds = \int_{\Gamma_j} n \cdot (\mu^L H^J)\, ds = \mu^L \int_{\Gamma_j} n \cdot \mathrm{curl}\, A\, ds = 0, \qquad j = 1,\ldots,m,$$

and

$$\int_{\Gamma_0} \tilde{f}\, ds = \int_{\Gamma_0} f\, ds - \int_{\Gamma_0} n \cdot (\mu^L H^J)\, ds = -\mu^L \int_{\Gamma_0} n \cdot \mathrm{curl}\, A\, ds = 0.$$

From these considerations we deduce

$$\begin{pmatrix} c \\ g \\ 0 \\ \tilde{f} \end{pmatrix} \in Y_d^{0\alpha}, \qquad < \begin{pmatrix} c \\ g \\ 0 \\ \tilde{f} \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ d' \\ 1 \end{pmatrix} > = 0,$$

and according to Theorem 4, the auxiliary problem (6)

$$\left.\begin{array}{l} \mathrm{curl}\, \tilde{H}^L = 0 \\ \mathrm{div}\, \tilde{H}^L = 0 \end{array}\right. \quad \text{in } G^L, \qquad \left.\begin{array}{l} (\Delta + k^2)\, H^E = 0 \\ k^2 = i\omega\sigma^E\mu^E \end{array}\right. \quad \text{in } G^E,$$

$$\left.\begin{array}{l} n \wedge H^E - n \wedge \tilde{H}^L = c \\ n \cdot (\mu^E H^E) - n \cdot (\mu^L \tilde{H}^L) = g \\ \mathrm{div}\, H^E = 0 \end{array}\right. \qquad \text{on } \Gamma,$$

$$n \cdot (\mu^L \tilde{H}^L) = \tilde{f} \qquad \text{on } \Gamma_0,$$

$$\int_{\gamma_i^L} \tau \cdot \tilde{H}^L\, dl = 0 \qquad i = 1,\ldots,p,$$

possesses a unique solution

$$\tilde{H}^L \in C^1(G^L) \cap C(\overline{G}^L),$$

$$H^E \in C^2(G^E) \cap C(\overline{G}^E), \quad \operatorname{div} H^E \in C(\overline{G}^E), \quad \operatorname{curl} H^E \in C(\overline{G}^E).$$

Since $\operatorname{div} H^E = 0$ on Γ and $\operatorname{Im}(k) > 0$, the divergence of H^E vanishes identically in G^E and $H^E, E^E = \frac{1}{\sigma^E} \operatorname{curl} H^E$ solve the time-harmonic Maxwell equations in G^E with coefficients μ^E, σ^E and ω.

On the other hand, for the field H^L,

$$H^L = \tilde{H}^L + H^J + H^Z \in C^1(G^L) \cap C(\overline{G}^L),$$

we get, using $\operatorname{curl} H^J = J_e$, $\operatorname{curl} H^Z = 0$,

$$\operatorname{curl} H^L = \operatorname{curl}(\tilde{H}^L + H^J + H^Z) = J_e \qquad \text{in } G^L.$$

Moreover

$$\left.\begin{aligned} n \wedge H^E &= n \wedge \tilde{H}^L + c = n \wedge (\tilde{H}^L + H^J + H^Z) = n \wedge H^L \\ n \cdot (\mu^E H^E) &= n \cdot (\mu^L \tilde{H}^L) + g = n \cdot (\mu^L(\tilde{H}^L + H^J + H^Z)) = n \cdot (\mu^L H^L) \end{aligned}\right\} \quad \text{on } \Gamma$$

and

$$\int_{\gamma_i^L} \tau \cdot H^L \, dl = \int_{\gamma_i^L} \tau \cdot (\tilde{H}^L + H^J + H^Z) \, dl = \int_{\gamma_i^L} \tau \cdot H^J \, dl + h_i^L - h_i^J = h_i^L, \qquad i = 1, \ldots, p.$$

Applying Stokes' Theorem we conclude

$$i\omega\mu^L \int_{\Gamma_j} n \cdot H^L ds = i\omega\mu^E \int_{\Gamma_j} n \cdot H^E ds = \mu^E \int_{\Gamma_j} n \cdot \operatorname{curl} E^E ds = 0, \qquad j = 1, \ldots, m,$$

so that Lemma 1 guarantees the existence of $E^L \in C^1(G^L) \cap C(\overline{G}^L)$ satisfying

$$\operatorname{curl} E^L = i\omega\mu^L H^L, \quad \operatorname{div} E^L = 0 \quad \text{in } G^L.$$

Therefore H^L, E^L, H^E, E^E solve (2) with prescribed circulations h_i^L for H^L.

■

With the help of this theorem, we finally obtain existence results, which are similar to those of the unbounded problem.

Corollary

For $J_e \in C^1(\mathbb{R}^3)$, $\operatorname{div} J_e = 0$, $\operatorname{supp}(J_e) \subset G^J$, $\overline{G}^J \subset G_0^L$ bounded, $f \in C^{0\alpha}(\Gamma_0)$, the bounded problem (2) is solvable if and only if

$$\int_{\Gamma_0} f \, ds = 0.$$

In the homogeneous case $J_e = 0$ we get exactly p linear independent solutions H^L, H^E, E^E, where p denotes the topological genus of G^E resp. G_0^L. The different solutions are characterized by their circulations

$$\int_{\gamma_i^L} \tau \cdot H^L \, dl = h_i^L, \qquad i = 1,\dots,p,$$

along γ_i^L.

E^L is not uniquely determined.

Proof

The first part follows directly from the last theorem by choosing the circulations h_i^L, $i = 1,\dots,p$ arbitrarily.

Using again Theorem 5, we get in the homogeneous case p solutions H_j^L, E_j^L, H_j^E, E_j^E, $j = 1,\dots,p$, of (2), having circulations $h_{ji}^L = \delta_{ij}$, $i = 1,\dots,p$. According to the uniqueness results from Theorem 2, these solutions are linear independent.

The nonuniqueness of E^L is obvious. ■

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